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DYNAMIC PROBLEMS ON THE BENDING OF A RECTANGULAR PLATE WITH MIXED FIXING CONDITIONS ON THE OUTLINE

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The dynamic problems on the bending of a rectangular slab whose lateral sides are hinge-fixed are studied. A mixed boundary condition is given on the upper face, and the lower face is a) hinge-fixed, b) clampled, c) simply supported. The problem of bending of a rectangular slab with initial conditions in the middle plane of the slab is studied analogously. A mixed boundary condition is given on the upper face, while the lateral and lower faces are hinge-fixed. The mixed boundary value problem is reduced to a conjugate series by separation of variables. Analogously to [1, 2], the conjugate series is reduced to a certain singular algebraic system of the first kind, which is then inverted exactly [3], and a new infinite algebraic system of the second kind is obtained. To overcome technical difficulties, a special approximation is introduced for the function in the first relationship in the conjugate series. The asymptotic properties of the infinite algebraic system obtained are studied. It is shown that the system is quasiregular for all the parameters in the problem. On the basis of the preceding, the possibility of raising the efficiency of the method is investigated and a domain of parameters for most efficient operation of the method is isolated for the problems being studied. The numerical material obtained verifies the high efficiency of the method in a quite extensive range of parameter variation. The results of a numerical analysis are presented in the form of graphs. The question of the eigenfrequencies of the problems posed is not examined.

1. Let us consider a rectangular plate of length $2b$ and height H. In conformity with the Kirchhoff theory the plate vibration equation has the form [4]

$$D\Delta\Delta w - \rho h \frac{\partial^2 w}{\partial t^2} = q(x, y, t) \tag{1.1}$$

Here $w(x, y, t)$ is the plate deflection, D is its bending stiffness, and $q(x, y, t)$ is the normal load. Furthermore, we assume that $q(x, y, t) = 0$. To investigate the harmonic oscillations of the plate, we seek the solution of (1.1) in the form

$$w(x, y, t) = w(x, y)\, e^{-i\omega t} \tag{1.2}$$

Let us examine three versions of the boundary conditions on the lower face of the plate [4]:

$$\text{a) } w(x, H) = w(x, 0) = M_y(x, 0) = 0 \quad (|x| \leqslant b) \tag{1.3}$$

$$M_y(x, H) = 0 \quad (-b \leqslant x < -a, \ a < x \leqslant b)$$

$$w_y'(x, H) = \theta(x) \quad (|x| \leqslant a)$$

(b) $w(x, H) = w(x, 0) = w_y'(x, 0) = 0 \quad (|x| \leqslant b)$
$M_y(x, H) = 0 \quad (-b \leqslant x < -a, \ a < x \leqslant b)$
$w_y'(x, H) = \theta(x) \quad (|x| \leqslant a)$
(c) $w(x, H) = M_y(x, 0) = Q_y(x, 0) = 0 \quad (|x| \leqslant b)$
$M_y(x, H) = 0 \quad (-b \leqslant x \leqslant -a, \ a \leqslant x \leqslant b)$
$w_y'(x, H) = \theta(x) \quad (|x| \leqslant a)$

Furthermore, for simplicity we have set $\theta(x) = \theta = \text{const}$. After substituting (1.2) into (1.1), we obtain an equation for the function $w = w(x, y)$

$$D\Delta\Delta w - \rho h \omega^2 w = 0$$

In order to satisfy the hinge-fixing on the lateral faces of the plate, we see the solution of this equation in the form

$$w(x, y) = \sum_{n=0}^{\infty} w_n(y) \cos\left[\frac{\pi(1+2n)}{2b} x\right] \tag{1.4}$$

Upon compliance with boundary conditions of the form a) — c), the mixed problem is reduced to the following conjugate series ($q(\xi)$ is the desired reactive moment at the support, and λ, β are dimensionless geometric parameters):

$$\sum_{n=0}^{\infty} q_n^\circ K(u_n h) \cos a u_n x = 1 \quad (|x| \leqslant a) \tag{1.5}$$

$$\sum_{n=0}^{\infty} q_n^\circ \cos a u_n x = 0 \quad (-b \leqslant x < -a, \ a < x \leqslant b)$$

$$u_n = \frac{\pi(1+2n)}{2b}, \quad q_n^\circ = \lambda \int_{-1}^{1} q^\circ(\xi) \cos a u_n \xi \, d\xi$$

$$q(\xi) = \beta(\lambda a)^{-1} D\theta q^\circ(\xi), \quad \lambda = a / b, \quad \beta = h / b$$

For the problems a) — c) the function $K(u)$ has the form

a) $K(u) = \varkappa^{-2} [\sigma_1 \operatorname{cth} \sigma_1 - \sigma_2 \operatorname{cth} \sigma_2]$ (1.6)

b) $K(u) = 2\varkappa^{-2} [\sigma_1\sigma_2 (1 - \operatorname{ch} \sigma_1 \operatorname{ch} \sigma_2) + u^2 \operatorname{sh}\sigma_1 \operatorname{sh} \sigma_2] R(u)$

$R(u) = [\sigma_1 \operatorname{ch} \sigma_1 \operatorname{sh} \sigma_1 - \sigma_2 \operatorname{ch} \sigma_2 \operatorname{sh} \sigma_1]^{-1}$

c) $K(u) = 2\varkappa^{-2} R(u) \{\sigma_1\sigma_2 [(1-\nu)^2 u^4 - \nu^4] +$
$u^2 \operatorname{sh} \sigma_1 \operatorname{sh} \sigma_2 [(1-\nu)^2 u^4 -$
$(1-2\nu)\varkappa^4] - \sigma_1\sigma_2 \operatorname{ch}\sigma_1 \operatorname{ch} \sigma_2 [(1-\nu)^2 u^4 + \varkappa^4]\}$

$$R(u) = \{\sigma_1 [(1-\nu) u^2 - \varkappa^2]^2 \operatorname{ch} \sigma_1 \operatorname{sh} \sigma_2 - \sigma_2 [(1-\nu) u^2 + \varkappa^2]^2 \operatorname{sh} \sigma_1 \operatorname{ch} \sigma_2\}^{-1}$$

$$(\sigma_{1,2} = \sqrt{u^2 \pm \varkappa^2}, \ \varkappa^2 = H^2 \omega \sqrt{\rho h D^{-1}})$$

where $\varkappa^2$ is the generalized dimensionless vibration frequency. The functions $K(u)$ of problems a) — c) are meromorphic in the complex plane $u = \sigma + i\tau$ and have zeros and poles on the real axis for definite values of $\varkappa$. An investigation of the function $K(u)$ shows that they are all even and possess identical asymptotic properties.

$$K(u) \sim A^\circ + O(u^2), \quad u \to 0 \quad (A^\circ = \lim K(u), \quad u \to 0) \qquad (1.7)$$
$$K(u) \sim u^{-1} + O(u^{-3}), \quad u \to \infty$$

The properties listed for the functions $K(u)$ afford an opportunity to represent it as the ratio of two entire functions

$$K(u) = A^\circ \frac{P_1(u^2)}{P_2(u^2)} = A^\circ \prod_{i=0}^{\infty} \frac{1 + u^2\delta_i^{-2}}{1 + u^2\gamma_i^{-2}} \qquad (1.8)$$

where A° is the same as in (1.7), $i\delta_n$ and $i\gamma_n$ are the countable quantities of zeroes and poles of the functions $K(u)$ in the upper half-plane and a finite number are on the real axis.

2. Let us also consider a rectangular plate of length $2b$ and height H with initial forces in the middle plane (problem d). The differential equation of plate bending vibrations has the form [4]

$$D\Delta\Delta w - \rho h \frac{\partial^2 w}{\partial t^2} = N_{11} \frac{\partial^2 w}{\partial x^2} + 2N_{12} \frac{\partial^2 w}{\partial x\, \partial y} + N_{22} \frac{\partial^2 w}{\partial y^2} + q \qquad (2.1)$$
$$\left(\frac{\partial N_{11}}{\partial x} + \frac{\partial N_{12}}{\partial y} = 0, \quad \frac{\partial N_{12}}{\partial x} + \frac{\partial N_{22}}{\partial y} = 0 \right)$$

Here N_{ij} are the forces acting in the plate middle plane which satisfy the equations in the parentheses, while w, q, D, ρ, h are the same as in (1.1).

Let us moreover assume that $q(x, y, t) = 0$. Since harmonic oscillations will henceforth be studied, the solution of (2.1) is naturally sought in the form (1.2). The boundary conditions have the form of conditions a) in (1.3).

As in Sect. 1, $\theta(x) = \theta = \text{const}$. We assume $N_{12} = 0$, and introduce the notation $N_{ii}D^{-1}2^{-1} = q_i^2$ $(i = 1, 2)$. After substituting (1.2) into (2.1), we obtain an equation for $w(x, y)$

$$D\Delta\Delta w - \rho h\omega^2 w = 2Dq_1^2 \frac{\partial^2 w}{\partial x^2} + 2Dq_2^2 \frac{\partial^2 w}{\partial y^2} \qquad (2.2)$$

Later $q_i = \text{const}$ everywhere.

Analogously to Sect. 1, we seek the solution of (2.2) in the form (1.4) in order to satisfy the hinge-fixing on the lateral faces of the plate.

By satisfying the boundary conditions a) in (1.3), we reduce the mixed problem to the solution of the conjugate series (1.5), where the function $K(u)$ has the form ($\varkappa$ is the same as in (1.6))

$$\text{d)} \quad K(u) = L^{-1/2} [\sigma_1 \operatorname{cth} \sigma_1 - \sigma_2 \operatorname{cth} \sigma_2]$$
$$\sigma_{1,2} = \sqrt{u^2 + q_2^2 \pm L^{1/2}}, \quad L = u^2(q_2^2 - q_1^2) + \varkappa^4 + q_2^4$$

We examine the case for q_i $(i = 1, 2)$ separately.

Let 1) $q_1 = q_2$, then the function $K(u)$ is meromorphic in the complex plane $u = \sigma + i\tau$, while in case 2) $q_2 > q_1$, the function $K(u)$ has branch points in the complex plane for $u = \pm iu^*$, and in case 3) $q_1 > q_2$, the branch points emerge on the real axis $u = \pm u^*$ $(u^* = [(\varkappa^4 + q_2^4)(q_1^2 - q_2^2)^{-1}]^{1/2})$.

An investigation of $K(u)$ shows that it is even and possesses the asymptotic properties (1.7).

Case 1) agrees completely with the cases considered in Sect. 1. In case 3), we make slits parallel to the imaginary axis, upward in the right half-plane and downward in the left, in the plane $u = \sigma + i\tau$. We fix the branch in the slit plane by the condition $\arg [K(u)] = 0$ as $u \to \infty$. In case 2), the branch points are connected by a slit through the infinitely remote point.

Moreover, the function $K(u)$ is approximated in an arbitrarily small neighborhood of the real axis by the function $K^*(u)$ which is meromorphic in the whole complex plane and has the asymptotic properties (1.7). Therefore, $K^*(u)$ can be represented in the form of (1.8).

3. Taking account of the properties of $K(u)$, the conjugate series (1.5) can be reduced to an infinite algebraic system of the form [1, 2]

$$CX = F \tag{3.1}$$

An irregular matrix A is separated out of the matrix C, and the system (3.1) takes the form (γ_m and δ_n are the same as in (1.8))

$$AX = BX + F \tag{3.2}$$

$$A = \{a_{nm}\} = \{(\gamma_m - \delta_n)^{-1}\}, \quad B = \{b_{mk}\}, \quad F = \{f_m\}, \quad X = \{x_n\}$$

$$b_{mk} = \frac{2\delta_k (c_m + d_k)}{(1 + c_m)(1 + d_k)(\gamma_m^2 - \delta_k^2)} \tag{3.3}$$

$$c_m = \exp(-2\gamma_m (1 - \lambda)\beta^{-1}), \quad d_k = \exp(-2\delta_k \lambda \beta^{-1})$$

$$f_m = -K^{-1}(0)\gamma_m^{-1}$$

Furthermore, inverting the matrix A exactly by using the Wiener — Hopf integral operator [1, 3] and the formula

$$\{(\gamma_m - \delta_n)^{-1}\} = \{[K_-^{-1}(\gamma_m)]' K_+'(-\delta_n)(\gamma_m - \delta_n)\}^{-1} \tag{3.4}$$

we arrive at a system of the second kind

$$X = A^{-1}BX + A^{-1}F \tag{3.5}$$

We digress here and investigate the system of the second kind obtained in [1, 2], since its asymptotic properties will be required later. After regularization, the infinite algebraic system in [1, 2] takes the form (3.5), where the relations

$$A^{-1} = \{\overset{\circ}{a}_{nm}\}, \quad \overset{\circ}{a}_{nm} = \frac{(2m-3)!!\,(2n-1)!!}{A^\circ (2m-2)!!\,(2n-2)!!\,(2n-2m+1)} \tag{3.6}$$

$$\gamma_m = \pi (A^\circ)^{-1}(m - 1/2), \quad \delta_k = \pi (A^\circ)^{-1} k \quad (k, m = 1, 2, \ldots)$$

hold for the approximation $K(u) \approx \operatorname{th}(A^\circ u)\, u^{-1}$ ($B = \{b_{nk}\}$ and $F = \{f_m\}$ exactly as in (3.5)).

Moreover, we investigate the matrix $A^{-1}B = G$. Its elements have the form

$$g_{nk} = \sum_{m=1}^{\infty} \overset{\circ}{a}_{nm} b_{mk}$$

The g_{nk} were investigated for large k by using the Euler — Maclauren summation formulas and the result was [5, 6]

$$g_{nk} \sim \frac{M_1}{k} + O\left(\frac{\ln k}{k\sqrt{k}}\right), \quad k \gg 1, \quad M_1 < \infty \tag{3.7}$$
$$g_{nk} \sim \frac{M_2}{\sqrt{n}} + O\left(\frac{\ln n}{n\sqrt{n}}\right), \quad n \gg 1, \quad M_2 < \infty$$
$$l_n \sim \frac{M_3}{\sqrt{n}} + O\left(\frac{\ln n}{n\sqrt{n}}\right), \quad A^{-1}F = \{l_k\}, \quad M_3 < \infty$$

The estimates (3.7) permit the assertion that an infinite algebraic system of the form (3.5) is quasiregular, and therefore, can be solved by the method of reduction [7].

We later investigate the solution of the conjugate series obtained in [1, 2]. It was written as

$$\frac{a}{D\theta} q(x) = \frac{\lambda}{\beta}\left[K^{-1}(0) + \sum_{n=1}^{\infty} x_n \operatorname{ch}\frac{a\delta_n x}{h} \operatorname{ch}^{-1}\frac{a\delta_n}{h}\right] \tag{3.8}$$

where x_k are the same as in (3.2) and (3.3). Let us analyze how the parameters in the series (3.8) influence its convergence. To do this, we write the n-th term of the series (3.8) and we convert it to the form

$$x_n \operatorname{ch}\frac{a\delta_n x}{h}\operatorname{ch}^{-1}\frac{a\delta_n}{h} \approx \frac{M_4}{\sqrt{n}}\exp\left(-\frac{\delta_n a}{h}(1-x)\right) + \tag{3.9}$$
$$O\left(\exp\left(-\frac{\delta_n a}{h}(1+x)\right)\right)$$
$$n \gg 1, \quad M_4 < \infty$$

by taking account of the last estimate in (3.7).

The deduction can be made that the radius of convergence of the series is $x < 1$. For $x = 1$ the series (3.8) diverges, but the last estimate in (3.7) affords the opportunity to conclude that $q(x)$ has a root singularity for $x = 1$. It must be noted (this follows from (3.9)) that the series will converge more rapidly, i.e., the method will perform much better, the greater the a / h and also if $\delta_n \gg 1$ $(n = 1, 2, \ldots)$. It turns out that if the approximation

$$K(u) = \frac{\operatorname{th} \varepsilon A^\circ u}{u}\prod_{i=1}^{N}\frac{u^2 + a_i^2}{u^2 + b_i^2}, \quad \varepsilon \prod_{i=1}^{N}\frac{a_i^2}{b_i^2} = 1, \quad \varepsilon \ll 1 \tag{3.10}$$

is introduced for the function $K(u)$, then the zeroes of $\operatorname{th} \varepsilon A^\circ u$, by which the solution is constructed, are very large compared to a_i in such an approximation, and the zeroes a_i will yield the main contribution to the solution.
Functions of the form

$$K_1(u) = \frac{\operatorname{sh} 2u - 2u}{2u \operatorname{sh}^2 u}, \quad K_2(u) = \frac{\operatorname{ch} 2u - 1 - 2u^2}{u(\operatorname{sh} 2u - 2u)}$$

can be approximated on the real axis by the expression (3.10) with $N = 3$, where $a_1 = 14.042$, $a_2 = 5.1949$, $a_3 = 80.217$, $b_1 = 28.045$, $b_2 = 2.4173$, $b_3 = 8.6316$, and $A^\circ = 0.6667$ for $K_1(u)$, while $a_1 = 1.7086$, $a_2 = 4.6173$, $a_3 = 66.597$, $b_1 = 11.778$, $b_2 = 2.6227$, $b_3 = 1.7008$ and $A^\circ = 0.5$ ($\varepsilon = 0.01$) for $K_2(u)$. The error in the approximation does not exceed 3%. An approximation method analogous to that used before was used here (*).

Numerical computations show that three equations of the infinite system are sufficient to acquire the accuracy needed (four significant figures) in the approximation (3.10) for the same value of a/h. It must be noted that the method will operate better, the greater the $a/h = \lambda/\beta$ (in the λ, β plane this is the domain where $\beta \leqslant \alpha\lambda$, $\alpha > 1$). Numerical computations confirm the mentioned deductions.

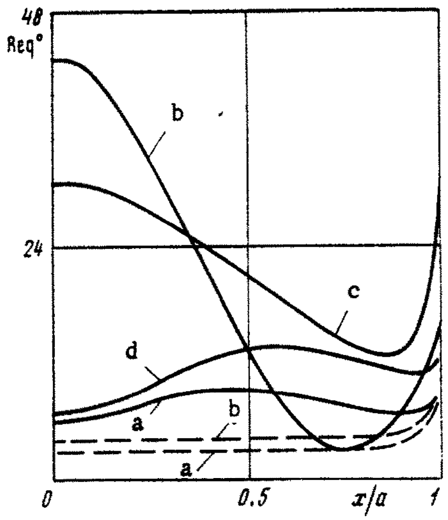

Fig. 1

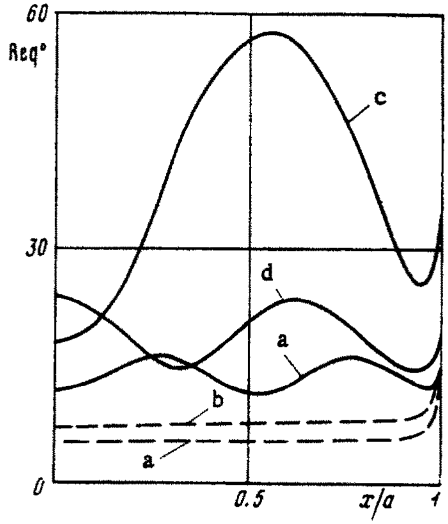

Fig. 2

The dashed lines in Figs. 1 and 2 represent graphs of the reactive moment at the support for $\lambda = 0.7$, $\beta = 0.25$ and $\lambda = 0.9$, $\beta = 0.25$, respectively, as evaluated by using the approximation (3.10) for $N = 3$.

4. For the numerical investigation of the problems posed in Sects. 1 and 2, the function $K(u)$ must be approximated, as before, by an expression with similar asymptotic properties which is easily factorizable. As noted above, zeroes and poles appear on the real axis at a definite generalized frequency $\varkappa$ in the function $K(u)$. Taking account of the experience in [8 — 10], we approximate the function $K(u)$ in the neighborhood of the real axis by an expression of the form

$$K(u) = \frac{\operatorname{th} \varepsilon A^\circ u}{u} \prod_{i=1}^{N_1} (u^2 - \zeta_i^2)^{-1} \prod_{i=1}^{N_2} (u^2 - \xi_i^2) \prod_{i=1}^{N_3} (u^2 + a_i^2) \prod_{i=1}^{N_4} (u^2 + b_i^2)^{-1} \tag{4.1}$$

$$A^\circ = \lim K(u), \quad u \to 0; \quad \varepsilon \ll 1$$

*) Zelentsov, V. B., Method of conjugate series - equations in the problem on the bending of a circular plate with mixed fixing conditions. Mixed Problems of the Mechanics of a Deformable Body. All-Union Scientific Conference. Abstracts of Reports, Pt. 2. Rostov-on-Don. 1977.

by extracting the zeroes and poles on the real axis.

As before, here ξ_i, ζ_i are the zeroes and poles on the real axis, a_i, b_i are approximation parameters, and N_3, N_4 are the quantities of approximation parameters. Clearly $N_1 + N_3 = N_2 + N_4$. Taking into account that the zeroes and poles of the approximation functions in the first relation of the conjugate series (1.5) have an identical asymptotic by number, it can be asserted that all the estimates (3.7) will be true to the accuracy of some constants for the system (3.5) as well. Therefore, the system (3.5) is quasiregular and can be solved by the method of reduction [7].

A numerical example was calculated for each problem a) – c). After having given the generalized frequency $\varkappa = 5$, approximations were found for the functions $K(u)$ of problems a) – d) for $\varepsilon = 0.01$, and are presented in the table.

An example for the problem d) was also examined for $q_1^2 = 5$, $q_2^2 = 0$ and $\varkappa = 5$. The zero and pole on the real axis are $\xi_1 = 3.196407$ and $\zeta_1 = 3.703872$. Comparing the ξ_1 (the phase velocities of the reactive moment) for problems a) and d) we see that the initial forces q_2 influence the phase velocities of the reactive moment more strongly.

Table 1

	a)	b)	c)	d)
A°	0.2592	0.08698	0.09565	0.1833
$N_1 = N_2$	1	1	2	1
$N_3 = N_4$	3	3	3	3
q_1^2	—	—	—	0
q_2^2	—	—	—	5
ξ_1	3.3732	2.0786	1.5863	2.7840
ξ_2	—	—	4.7454	—
ζ_1	3.8898	3.3732	2.7352	3.4283
ζ_2	—	—	4.8167	—
a_1	95.009	179.17	326910	224.48
a_2	0.6773	1.0875	15.598	0.6773
a_3	3.0166	4.3380	6.9051	3.0166
b_1	18.296	2.9641	5.3570	1.9296
b_2	0.7771	1.1968	6.8634	0.7771
b_3	11.847	14.682	54729	24.794

All the approximations have been obtained with less than 3% error on the whole real axis. Keeping all the remarks made in Sect. 3 relative to the efficiency of the method in mind, for the approximations made here it is possible to limit oneself to $N_2 + N_3$ equations of the infinite system in order to achieved the needed accuracy in a broad range of variation of the parameters λ, β. This suggests that a system of moderate order should certainly not be inverted exactly by means of (3.4) and (3.5) (this last results in awkward formulas for the elements of the matrix A^{-1}), but it can be inverted numerically on an computer. Results of computations did not disclose any discrepancy in accuracy between the numerical and exact inversions in all the problems solved here.

The solution of the conjugate series (1. 5) for $q(x)$ is given by formula (3. 8), where

$$\delta_n = \{a_1,\ a_2,\ a_3,\ i\xi_l,\ \pi n (A^\circ\varepsilon)^{-1}\},\ |x| \leqslant 1 \tag{4. 2}$$

and $l = 1$ for problems a), b), d) and $l = 2$ for problem c). The solid lines, presented as an illustration in Fig. 2 and 3, are graphs of the changes in $\operatorname{Re} q(x)$ for problems a) — d), respectively, for $\lambda = 0.7, \beta = 0.25$ and $\lambda = 0.9, \beta = 0.25$. The singularity in $q(x)$ at $x = 1$ can be isolated similarly [11]. However, there is no special need for this since the singularity only appears very near the edge of the support. This easily noted in an analysis of (4. 1) and (4. 2).

Attention is turned to the fact that in mechanically more stiff systems the zeroes and poles emerge later on the real axis than in less stiff systems as the generalized frequency increases. Use of the Kirchhoff —Love model equations for higher generalized vibration frequencies ($\varkappa > 5$) can raise some doubts. Hence, it is more expedient to use more exact applied models of slab bending [4] at high frequencies.

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